The Fundamental Theorem of Calculus

Calculus

Derivatives $\xleftrightarrow{\text{Fundamental Theorem}}$ Definite Integrals

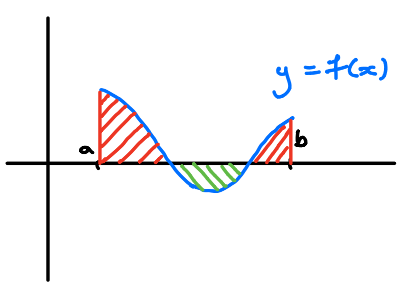
Slopes of Tangent Lines — Net Areas enclosed by curves

The Fundamental Theorem of Calculus :

Continuous on $[a,b]$ — Notation

$$\int f(x)\,dx = F(x) + C \implies \int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b$$

Conclusion : To calculate a definite integral we should first calculate an indefinite integral.



$$\Rightarrow \int_a^b f(x)\,dx = \text{Area}(\;) - \text{Area}(\;)$$

$$= F(b) - F(a)$$

Fundamental Theorem

Examples

1/ $\displaystyle\int_0^1 x^2\,dx = ?$

$$\int x^2\,dx = \frac{1}{3}x^3 + C \implies \int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}1^3 - \frac{1}{3}0^3 = \frac{1}{3}$$

2/ $\displaystyle\int_{-1}^{2} x\,2^{(x^2)}\,dx = ?$

$$u = x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$$

$$\Rightarrow \int x\, 2^{(x^2)}\, dx = \int x\, 2^{(x^2)} \cdot \frac{du}{2x} = \int \frac{1}{2} \cdot 2^u\, du = \frac{1}{2\ln(2)} 2^u + C$$

$$= \frac{1}{2\ln(2)} 2^{(x^2)} + C$$ ← Very important to switch back to x before calculating definite integral

$$\Rightarrow \int_{-1}^{2} x\, 2^{(x^2)}\, dx = \frac{1}{2\ln(2)} 2^{(x^2)} \Big|_{-1}^{2} = \frac{1}{2\ln(2)} 2^4 - \frac{1}{2\ln(2)} 2$$

$$= \frac{7}{\ln(2)}$$

3/ A partical is travelling in a straight line with velocity $v(t) = t^3 - 1$. What is the <u>net</u> distance travelled between $t = 0$ and $t = 2$? What is the <u>total</u> distance travelled between $t = 0$ and 2?

General observation 1

$s(t)$ = position at time t $\Rightarrow$ $s'(t) = v(t)$

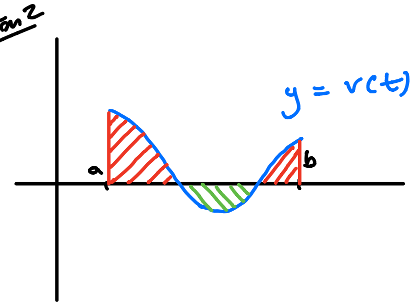
$$\Rightarrow \int_a^b v(t)\, dt = s(t)\Big|_a^b = s(b) - s(a)$$ = Net distance travelled between $t = a$ and $t = b$

$$\int t^3 - 1\, dt = \frac{1}{4}t^4 - t + C \Rightarrow \int_0^2 t^3 - 1\, dt = \frac{1}{4}t^4 - t \Big|_0^2$$

$$= 2$$

$\Rightarrow$ Net distance travelled between $t = 0$ and $t = 2$ is 2.

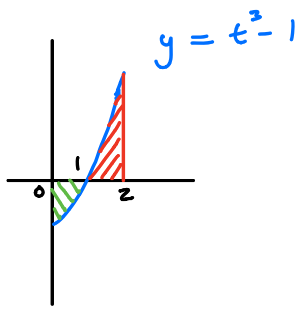
General observation 2



Area(red) = Distance travelled to right.

$\Rightarrow$ Area(green) = Distance travelled to left

Area(red) + Area(green) = Total distance travelled



$$\text{Area (red)} = \int_1^2 t^3 - 1 \, dt = \frac{1}{4}t^4 - t \Big|_1^2 = 2 - \left(\frac{1}{4} - 1\right)$$
$$= 2 + \frac{3}{4} = \frac{11}{4}$$

$$\text{Area (green)} = -\int_0^1 t^3 - 1 \, dt = -\frac{1}{4}t^4 + t \Big|_0^1$$
$$= \left(-\frac{1}{4} + 1\right) - 0$$
$$= \frac{3}{4}$$

Definite integral counts regions under x-axis negatively

$\Rightarrow$ Total distance travelled between $t = 0$ and 2 $= \frac{11}{4} + \frac{3}{4} = \frac{14}{4}$

4/ The marginal cost of a product is $x^2 + x + 1$. (number of units made)
What is the increase to costs when changing production from 1 to 3?

$C(x)$ = Cost of making x units.

$$\Rightarrow \quad C'(x) = x^2 + x + 1$$

Fundamental Theorem

$$\Rightarrow \int_1^3 x^2 + x + 1 \, dx = C(3) - C(1)$$ = Increase to costs changing from $x = 1$ to $x = 3$.

$$\int x^2 + x + 1 = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + C$$

$$\Rightarrow \int_1^3 x^2 + x + 1 = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x \Big|_1^3 = \left(\frac{1}{3} \cdot 3^3 + \frac{1}{2}3^2 + 3\right) - \left(\frac{1}{3} + \frac{1}{2} + 1\right)$$
$$= \frac{88}{6}.$$

5/ $$\int_{-2}^{1} \frac{1}{x^2} \, dx = \, ?$$

WRONG: $$\int \frac{1}{x^2} dx = \frac{-1}{x} + C \Rightarrow \int_{-2}^{1} \frac{1}{x^2} dx = \frac{-1}{x} \Big|_{-2}^{1}$$
$$= \frac{-1}{1} - \frac{-1}{-2} = -\frac{3}{2}$$

This can't be correct because $\frac{1}{x^2} > 0$ on $[-2, 1]$.

Problem : $\frac{1}{x^2}$ has a vertical asymptote at $x = 0$ in $[-2, 1]$

$\Rightarrow \displaystyle\int_{-2}^{1} \frac{1}{x^2}\,dx$ does not have meaning.

<u>Remark</u> The only major drawback of using the Fundamental Theorem is that finding an antiderivative may be very difficult. For example we cannot calculate $\displaystyle\int_0^1 e^{(x^2)}\,dx$.

General Properties of Definite Integral

$$\int_a^b k f(x)\,dx = k \cdot \int_a^b f(x)\,dx$$

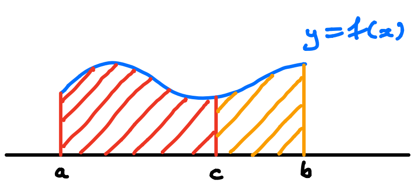
$$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Can use this to break an integral into easier pieces

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$



$\Rightarrow$ Area under curve between a and b = Area ( ) + Area ( )